1. Complex Numbers
   1. Algebra of Complex Numbers
      1. Arithmeric Operations
         1. imaginary unit i : i^{2} = -1
            1. i^{n} has only 4 possible values : 1, i, -1, -i
         2. complex number \alpha + i \beta : two real numbers \alpha and \beta
            1. \alpha and \beta are the real and imaginary part of the complex number
            2. if \alpha = 0, the number is purely imaginary
            3. if \beta = 0, the number is real.
            4. Zero : Both purely imaginary and real
         3. Two complex numbers are equal iff they have the same real part and same imaginary part
         4. Addition (\alpha + i \beta) + (\gamma + i \delta) = (\alpha + \gamma) + i (\beta + \gamma)
         5. Multiplication (\alpha + i \beta) (\gamma + i \delta) = (\alpha \gamma - \beta \delta) + i(\alpha \delta + \beta \gamma)
         6. Division (\alpha + i \beta) / (\gamma + i \delta) is a complex number.
            1. \frac{\alpha + i \beta}{\gamma + i \delta} = \frac{\alpha \gamma + \beta \delta}{ \gamma^{2} + \delta^{2} } + i \frac{\beta \gamma - \alpha \delta}{ \gamma^{2} + \delta^{2} }
         7. reciprocal of a complex number \alpha + i \beta \neq 0 : \frac{1}{\alpha + i \beta} = \frac{\alpha – i \beta}{\alpha^{2} + \beta^{2}}
      2. Square Roots
         1. The square root of a complex number \sqrt{\alpha + i \beta} exists
            1. if \beta \neq 0, \sqrt{\alpha + i \beta} = \mp \biggl( \sqrt{ \frac{\alpha + \sqrt{\alpha^{2} + \beta^{2}}}{2} } + i \frac{\beta}{|\beta|} sqrt{\frac{- \alpha + \sqrt{\alpha^{2} + \beta^{2}}}{2} } \biggr)
            2. else if \beta = 0 \begin{cases} \mp \sqrt{\alpha} & \alpha \ge 0 \\ \mpi|sqrt{-\alpha} & \alpha < 0\end{cases}
      3. Justification
         1. \mathbf{R} is field.
            1. addition, multiplication is defined, satisfying the associative, commutative, distributive laws.
            2. 0, 1 are neutral elements under addition and multiplication, respectively. equation of subtraction \beta + x = \alpha always has a solution. equation of division \beta x = \alpha has a solution whenever \beta \neq 0.
            3. neutral elements and the result of subtraction and division are unique.
         2. every field is an integral domain : \alpha \beta = 0 iff \alpha = 0 or \beta = 0
         3. The field \mathbf{R} has an order relation \alpha < \beta. \alpha < \beta iff \beta - \alpha \in \mathbf{R^{+}}
            1. The set \mathbf{R^{+}} of positive numbers

0 is not positive number

if \alpha \neq 0 either \alpha or -\alpha is positive

Sum and the product of two positive numbers are positive

* + - 1. \mathbf{R} contains natural numbers and contain the subfield formed by all rational numbers
      2. \mathbf{R} satisfies completeness condition : every increasing and bounded sequence of real numbers has a limit.
         1. Let \alpha\_{1} < \alpha\_{2} < …< \alpha\_{n} < … and assume the existence of a real number B s.t. \alpha\_{n} for all n. Then exists a number A = \lim\_{n \to \infty}\alpha\_{n} with following property:

Given any \epsilon >0 there exists a natural number n\_{0} s.t. A - \epsilon < \alpha\_{n} < A for all n > n\_{0}.

* + - 1. Suppose that a field \mathbf{F} can be found which contains \mathbf{R} as a subfield, and in which the equation x^{2} + 1= 0 can be solved. Denote a solution by i.
      2. Let \mathbf{C} be the subset of \mathbf{F} consisting of all elements which can be expressed in the form \alpha + i \beta with real \alpha and \beta.
         1. The representation is unique.
         2. \mathbf{C} is a subfield of \mathbf{F}. The structure of \mathbf{C} is independent of \mathbf{F}
      3. Define the field of complex numbers to be the subfield \mathbf{C} of an arbitrary given \mathbf{F}.
         1. There exists a field \mathbf{F} which contains a subfield isomorphic with \mathbf{R} and which the equation x^{2} + 1 = 0 has a root.
    1. Conjugation, Absolute Value
       1. Real and Imaginary part of a complex number a will be denoted by Re a, Im a.
       2. Transformation which replaces \alpha + i \beta to \alpha – i \beta is called complex conjugation.
          1. \alpha - i \beta is the conjugate of \alpha + i \beta.
          2. Conjugate of a is denoted by \bar{a}.
          3. Conjugation is an involutory transformation : \bar{\bar{a}} = a
          4. Re a = \frac{a + \bar{a}}{2}, Im a = \frac{a - \bar{a}}{2i}
          5. \bar{a+b} = \bar{a} + \bar{b} , \bar{ab} = \bar{a} \cdot \bar{b}
          6. Let R(a,b,c,…) stand for any rational operation applied to the complex numbers a,b,c, … Then \bar{R(a,b,c,….)} = R(\bar{a},\bar{b}.\bar{c},….)
          7. nonreal roots of an equation with real coefficients occur in pairs of conjugate roots.
       3. Product a\bar{a} = \alpha^{2} + \beta^{2} is always positive or zero. Its nonnegative square root is called absolute value of the complex number a.
          1. It is denoted by |a|.
          2. a\bar{a} = |a|^{2}, and |ab| = |a| \cdot |b|
          3. For arbitrary finite products, |a\_{1}a\_{2} \cdots a\_{n}| = |a\_{1}| \cdot |a\_{2}| \cdots |a\_{n}|
          4. if b \neq 0, quotient |\frac{a}{b}| = \frac{|a|}{|b|}
          5. |a + b|^{2} = |a|^{2} + |b^{2}| + 2 Re a\bar{b}
          6. |a - b|^{2} = |a|^{2} + |b^{2}| - 2 Re a\bar{b}
          7. |a + b|^{2} + |a - b|^{2} = 2( |a|^{2} + |b^{2}| )
    2. Inequalities
       1. –|a| \le Re a \le |a|, -|a| \le Im a \le |a|
          1. Re a = |a| holds iff a is real and \ge 0.
       2. (Triangle inequality) |a+b| \le |a| + |b|
       3. Arbitrary sums |a\_{1} + a\_{2} + …. + a\_{n}| \le |a\_{1}| + |a\_{2}| + …. + |a\_{n}|
          1. equality holds iff the ratio of any two nonzero terms is positive.
       4. |a| - |b| \le |a-b| , |a-b| \ge ||a|-|b||
       5. |\alpha + i \beta| \le |\alpha| + |\beta|
       6. Cauchy’s inequality |\sum\_{i = 1}^{n} a\_{i}b\_{i}|^{2} \le \sum\_[i=1]^{n}|a\_{i}|^{2} \sum\_[i=1]^{n}|b\_{i}|^{2}
  1. Geometric Representations of Complex Numbers
     1. a = \alpha + i\beta can be represented by points with coordinates (\alpha,\beta)
        1. first coordinate axis (x-axis) is real axis
        2. second coordinate axis (y-axis) is imaginary axis.
        3. plane itself is called complex plane.
     2. Geometric Addition and Multiplication
        1. Addition of complex numbers can be visualized as vector addition.
        2. Polar coordinates : if the polar coordinates of the point (\alpha, \beta) are ( r, \phi), \alpha = r cos \phi , \beta = r sin \phi
           1. In this Trigonometric form of a complex number r is always \ge 0 and equal to the modulus |a|.
           2. Polar angel \phi is called the argument or amplitude of the complex number, denoted by arg a.
        3. Two complex numbers a\_{1} = r\_{1}(cos \phi\_{1} + i sin \phi\_{1}) and a\_{2} = r\_{2}(cos \phi\_{2} + i sin \phi\_{2}). a\_{1}a\_{2} = r\_{1}r\_{2}[cos( \phi\_{1} + \phi\_{2} ) + i sin(\phi\_{1} + \phi\_{2})].
           1. argument of a product is equal to the sum of the arguments of the factors.
     3. Binomial equation
        1. Powers of a = r(cos \phi + i sin \phi) : a^{n} = r(cos n\phi + i sin n\phi).
        2. De Moivre’s formula (cos \phi + i sin \phi)^{n} = cos n\phi + i sin n\phi
        3. nth root of a complex number a supposing that a \neq 0 and a = r(cos \phi + i sin \phi) : z^{n} = a \implies z = \sqrt[n]{r} \biggl[ cos \biggl( \fact{\phi}{n} + k \fact{2\pi}{n} \biggr) + i sin \biggl( fact{\phi}{n} + k \fact{2\pi}{n} \biggr ) \biggr] , k = 0, 1, …., n-1
           1. There are n nth roots of any complex number \neq 0.
           2. Geometrically, nth root are the vertices of a regular polygon with n sides.
        4. Root of the equation z^{n} = 1 are called nth roots of unity.
           1. \omega = cos \frac{2\pi}{n} + i sin \frac{2\pi}{n}, all the roots are 1, \omega, \omega^{2}, …., \omega^{n-1}

If \sqrt[n]{a} denotes any nth root of a, then all the nth roots can be expressed in the form \omega^{k} \cdot \sqrt[n]{a}, k = 0, 1, …, n-1

* + 1. Analytic Geometry
       1. Complex equation is equivalent to two real equations.
       2. Straight line in the complex plane is given by a parametric equation z = a + bt, where a and b are complex numbers and b\neq 0; the parameter t runs through all real values.
          1. Two equations z = a + bt and z = a’ + b’t is

same line iff a’-a and b’ are real multiples of b

parallel if b’ is a real multiple of b

equally directed if b’ is a positive multiple of b

angle between them is arg b’/b

orthogonal to each other if b’/b is purely imaginary

* + - * 1. a directed line a+bt

right half plane with Im (z-a)/b <0 and a left half plane with Im(z-a)/b >0

* + - 1. inequality |z-a| < r describes the inside of a circle
    1. Spherical Representation
       1. Extend the system \mathbf{C} of complex numbers by an introduction of a symbol \infty
          1. a + \infty = \infty + a = \infty for all finite a
          2. b \cdot \infty = \infty \cdot b = \infty for all b \neq 0, including b = \infty.
          3. a / 0 = \infty for a \neq 0 and b / \infty = 0 for b \neq \infty
       2. Extended Complex plane : The points in the plane together with the point at infinity
       3. Unit Sphere S : equation in three dimensional space is x\_{1}^{2} + x\_{2}^{2} + x\_{3}^{2} = 1
          1. With every point on S except (0,0,1), associate a complex number z = \frac{x\_{1} + i x\_{2}}{1 – x\_{3}}
          2. Let the point at infinity correspond to (0,0,1).

Hemisphere x\_{3} < 0 corresponds to the disk |z| < 1

Hemisphere x\_{3} > 0 to |z| > 1.

* + - * 1. In function theory S is referred to as the Riemann Sphere.
      1. For z = x + i y , the points (x,y,0) (x\_{1}, x\_{2},x\_{3}-1) and (0,0,1) are in a straight line. The correspondence is a central projection from the center (0,0,1), it is called a stereographic projection.
         1. Any circle on the sphere corresponds to a circle or straight line in the z – plane.
      2. distance d(z,z’) between the stereographic projections of z and z’ : d(z,z’) = \frac{2|z-z’|}{\sqrt{1 + |z|^{2}}}

1. Complex Functions
   1. Introduction to the concept of Analytic function
      1. Notation for a complex function of a complex variable : w = f(z)
         1. a general function of a real or complex variable : y = f(x)
      2. every function is defined on an open set, by which we mean that if f(a) is defined, then f(x) is defined for all x sufficiently close to a.
      3. Limits and Continuity
         1. The function f(x) is said to have the limit A as x tends to a, \lim\_{x \to a} f(x) = A iff the following is true. Denote as f(x) \to A for x \to A.
            1. For every \epsilon >0 there exists a number \delta >0 with the property that |f(x) – A| < \epsilon for all values of x s.t. |x-a| < \delta and x \neq a.
         2. (properties)
            1. Results concerning the limit of a sum, a product, a quotient hold in the complex case.
            2. \lim\_{x \to a}\bar{ f(x)} = \bar{A}
            3. \lim\_{x \to a} Re f(x) = Re A \lim\_{x \to a} Im f(x) = Im A
         3. The function f(x) is said to be continuous at a iff \lim\_{x \to a} f(x) = f(a).
         4. Continuous function is one which is continuous at all points where it is defined.
            1. the sum f(x) + g(x) and the product f(x)g(x) of two continuous functions are continuous.
            2. the quotient f(x)/g(x) is defined and continuous at a iff g(a) \neq 0.
            3. If f(x) is continuous, so are Re f(x), Im f(x), |f(x)|.
         5. Derivative of a function f : f’(a) = \lim\_{x\to a} \frac{f(x) – f(a)}{x – a}.
            1. Usual rules for forming the derivative of a sum, a product, or a quotient are all valid. The derivative of a composite function is determined by the chain rule.
            2. Real function of a complex variable either has the derivative zero, or else the derivative does not exist.
            3. Let z(t) = x(t) + i y(t). z’(t) = x’(t) + i y’(t) and the existence of z’(t) is equivalent to the simultaneous existence of x’(t) and y’(t).
      4. Analytic function
         1. Class of Analytic functions is formed by the complex functions of a complex variable which possess a derivative wherever the function is defined.
            1. Also called holomorphic function.
            2. The sum and the product of two analytic functions are analytic.
            3. quotient of two analytic functions is analytic provided that g(z) does not vanish.
            4. If a function is analytic, it is necessarily continuous.
         2. Let f(z) = u(z) = i v(z), z = x + i y. the following equations are the Cauchy-Riemann differential equations which must be satisfied by the real and imaginary part of any analytic functions.
            1. Real equations \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}.
         3. (properties)
            1. Simplest equation for f’(z) : f’(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
            2. |f’(z)|^{2} = (\frac{\partial u}{\partial x})^{2} + (\frac{\partial u}{\partial y})^{2} = (\frac{\partial u}{\partial x})^{2} + (\frac{\partial v}{\partial x})^{2} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}
            3. derivative of an analytic function is analytic, so u and v will have continuous partial derivatives of all order, and in particular the mixed derivatives will be equal. Therefore,

\Delta u = \frac{\partial^{2} u}{\partial x^{2}} + \frac{\partial^{2} u}{\partial y^{2}} = 0

\Delta v = \frac{\partial^{2} v}{\partial x^{2}} \frac{\partial^{2} v}{\partial y^{2}} = 0

* + - 1. A function u which satisfies Laplace’s equation \Delta u = 0 is said to be harmonic.
         1. Real and imaginary part of an analytic function are harmonic.
      2. If two harmonic functions u and v satisfy the Cauchy-Riemann equations, then v is said to be the conjugate harmonic function of u.
         1. Then u is the conjugate harmonic function of -v.
      3. If u(x,y) and v(x,y) have continuous first-order partial derivatives which satisfy the Cauchy-Riemann differential equations, then f(z) = u(z) + i v(z) is analytic with continuous derivative f’(z), and conversely.
      4. Let a complex function f(x,y) of two real variables. Introduce a complex variable z = x + iy and its conjugate \bar{z} = x – iy. If f is analytic, \frac{\partial f}{\partial \bar{z}} = 0, so f is independent of \bar{z}, and a function of z alone.
      5. Simple method to compute the analytic function f(z) whose real part is a given harmonic function u(x,y) : f(z) = 2u(z/2,z/2i) – u(0,0). a purely imaginary constant can be added at will.
    1. Polynomials
       1. Every constant is an analytic function with derivative 0.
          1. a polynomial constant 0 is excluded from our consideration.
       2. Simplest nonconstant analytic function z is derivative 1.
       3. Every polynomial P(z) = a\_{0} + a\_{1} z + … + a\_{n} z^{n} is an analytic function.
       4. (Fundamental Theorem of Algebra)For n > 0 the equation P(z) = 0 has at least one root.
          1. P(z) = a\_{n}(z-\alpha\_{1})(z-\alpha\_{2})\cdots(z-\alpha\_{n}), where \alpha\_{1}, \alpha\_{2}, … , \alpha\_{n} are not necessarily distinct.

If exactly h of the \alpha\_{i} coincide, their common value is called a zero of P(z) of the order h.

for that \alpha, P(\alpha) = P’(\alpha) = \cdots = P^{h-1} (\alpha) = 0 while P^h(\alpha) \neq 0.

a zero of order 1 is called a simple zero.

* + - 1. (Lucas’s Theorem) (Thm 2.1) If all zeros of a polynomial P(z) lie in a half plane, then all zeros of the derivative P’(z) lie in the same half plane.
         1. the smallest convex polygon that contains the zeors of P(z) also contains the zeros of P’(z).
    1. Rational Functions
       1. a Rational function R(z) = \frac{P(z)}{Q(z)} given as the quotient of two polynomials.
          1. Assume P(z) and Q(z) has no common zeros.
          2. R(z) must be considered as a function with values in the extended plane.
          3. Zeros of Q(z) are called poles of R(z), and the order of a pole is equal to the otder of the corresponding zero of Q(z).
       2. Common number of zeros and poles is called the order of the rational function.
       3. For R(z) = \frac{a\_{0} + a\_{1}z + \cdots a\_{n}z^{n}}{b\_{0} + b\_{1}z + \cdots b\_{m}z^{m}} Number of zeros is equal to or greater of the numbers m and n.
       4. A rational function R(z) of order p has p zeros and p poles, and every equation R(z) = a has exactly p roots.
       5. linear transformation z + a is called a parallel translation, and 1/z is an inversion.
       6. Every rational function has a representation by partial fractions.
          1. R(z) = G(z) + H(z) where G(z) is a polynomial without constant term, and H(z) is finite at \infty.
          2. degree of G(z) is the order of pole at \infty, and the polynomial G(z) is called the singular part of R(z) at \infty.
       7. Let the distinct finite poles of R(z) be denoted by \beta\_{1}, \beta\_{2}, …, \beta\_{q}. then R(z) = G(z) + \sum\_{j = 1}^{q} G\_{j} (\frac{1}{z-\beta\_{j}}).
  1. Elementary Theory of Power Series
     1. Sequences
        1. Sequence {a\_{n}}\_{1}^{\infty} has the limit A if to every \epsilon >0 there exists an n\_{0} s.t. |a\_{n} – A| < \epsilon for n \ge n\_{0}.
           1. Sequence with a finite limit is said to be convergent.
           2. Any sequence which does not converge is divergent.
           3. If \lim\_{n\to\infty} a\_{n} = \infty, the sequence is said to diverge to infinity.
        2. A sequence will be called fundamental, or a Cauchy sequence if it satisfies the following condition : given any \epsilon > 0 there exists an n\_{0} s.t. |a\_{n} – a\_{m}| < \epsilon whenever n \ge n\_{0} and m \ge n\_{0}.
           1. A sequence if convergent iff it is a Cauchy sequence.
        3. Given a real sequence {\alpha\_{n}}\_{1}^{\infty} set a\_{n} = max{\alpha\_{1}, …, \alpha\_{n}}. The limit A\_{1} of a sequence {a\_{n}}\_{1}^{\infty} is known as the limit point or supremum of the numbers \alpha\_{n}.
           1. denoted l.u.b or sup
           2. Construct the least upper bound A\_{k} for the sequence {a\_{n}}\_{k}^{\infty}. Write the limit of the sequence {A\_{k}} A = lim\_{n \to \infty} sup \alpha\_{n}. A is called the limes superior.

There are arbitrarily large n for which \alpha\_{n} > A - \epsilon.

denoted \overline{\lim}. limes inferior denoted as \underline{\lim}.

* + - * 1. \underline{\lim} \alpha\_{n}+ \underline{\lim} \beta\_{n} \underline{\lim} (\alpha\_{n} + \beta\_{n} ) \underline{\lim} \alpha\_{n} \overline{\lim} .\beta\_{n}
        2. \underline{\lim} \alpha\_{n} + \overline{\lim}.\beta\_{n} \overline{\lim} (\alpha\_{n} + \beta\_{n}) \overline{\lim}. \alpha\_{n} + \overline{\lim}. \beta\_{n}
    1. Series
       1. If |b\_{m} – b\_{n}| \le |a\_{m} – a\_{n}| for all pairs of subscripts, the sequence {b\_{n}} is a contraction of the sequence {a\_{n}} . (nonstandard term)
       2. An infinite series is a formal infinite sum a\_{1} + a\_{2} + … + a\_{n} + .. associated with the sequence of its partial sums s\_{n} = a\_{1} + a\_{2} + … + a\_{n}.
          1. The series is said to converge iff the corresponding sequence is convergent. The limit of the sequence is the sum of the series.
       3. The series formed by absolute values of the terms |a\_{1}| + |a\_{2} |+ … + |a\_{n}| + …
          1. The sequence of partial sums of original series is a contraction of the sequence corresponding to the series formed by absolute values.
          2. Convergent of the series formed by absolute values implies that the original series is convergent.
          3. A series with the property that the series formed by absolute values converges is said to be absolutely convergent.
    2. Power series
       1. Power series is of the form a\_{0} + a\_{1}z + a\_{2}z^{2} + … + a\_{n}z^{n} + … where the coefficients a\_{n} and the variable z are complex.
          1. series \sum\_{n = 0}^{\infty} a\_{n}(z – z\_{0})^{n} which are power series with respect to center z\_{0}
          2. geometric series 1 + z + z^{2} + … + z^{n} +..
       2. (Thm 2.2) For every power series a\_{0} + a\_{1}z + a\_{2}z^{2} + … + a\_{n}z^{n} + … there exists a number R, 0 \le R \e \infty, called the radius of convergence, with the following properties:
          1. The series converges absolutely for every z with |z| < R. If 0 \le \rho < R the convergence is uniform for |z| \le \rho.
          2. If |z| > R the terms of the series are unbounded, and the series is consequently divergent.
          3. In |z| < R the sum of the series is an analytic function. The derivative can be obtained by term wise differentiation, and the derived series has the same radius of convergence.
       3. The circle |z| < R is called the circle of convergence
       4. (Hadamard’s formula for the radius of convergence) 1/R = \lim\_{n \to \infty} sup \sqrt[n]{|a\_{n}|}.
       5. A power series with positive radius of convergence has derivatives of all orders.
          1. a\_{k} = f^{k}(0) /k!, and the power series becomes f(z) = f(0) + f’(0) z + \frac{f’’(0)}{2!} + … + \frac{f^{n}(0)}{n!}z^{n} + … . This is a Taylor-Maclaurin development.
    3. Abel’s Limit theorem
       1. Assume R = 1 and the convergence takes place at z = 1.
       2. (Thm 2.3) If \sum\_{0}^{\infty} a\_{n} converges, then f(z) = \sum\_{0}^{\infty} a\_{n}z^{n} tends to f(a) as z approaches 1 in such a way that |1-z|/(1-|z|) remains bounded.
          1. z stays in an angle < 180 with vertex 1 , symmetrically to the part (-\infty, 1) of the real axis. it is said the approach takes place in a Stolz angle.

1. Analytic functions as mappings
   1. Conformality
      1. Arcs and Closed Curves
         1. Equation of an arc \gamma in the plane is given in parametric form x = x(t), y = y(t) where t runs through an interval \alpha \le t \le \beta and x(t), y(t) are continuous functions.
            1. complex notation z = z(t) = x(t) + i y(t)
            2. \gamma is a continuous mapping of [\alpha, \beta] and in this case denote z = \gamma (t).
            3. as a point set arc is compact and connected.
         2. Let a nondecreasing function t = \phi(\tau) maps interval \alpha’ \le \tau \le \beta’ onto \alpha \le t \le \beta, then z = z(\phi(\tau)) defines the same succession of points of arc z = z(t). It is done by a change of parameter.
            1. Change is reversible iff \phi(\tau) is strictly increasing.
            2. linear change of parameter : t = a \tau + b, a >0
            3. initial point and terminal point of an arc remain the same after a change of parameter.
         3. If the derivative z’(t) = x’(t) + i y’(t) exists and is \neq 0, the arc \gamma has a tangent whose direction is determined by arg z’(t).
         4. Arc is differentiable if z’(t) exists and is continuous.
            1. If, in addition, z’(t) \neq 0 the arc is said to be regular.
            2. Arc is said to be piecewise differentiable or piecewise regular if the same conditions hold except for a finite number of values t.
            3. Differentiable or regular character of an arc is invariant under the change of parameter t = \phi(\tau) provided that \phi’(\tau) is continuous and, for regularity, \neq 0.
         5. Arc is simple, or a Jordan arc, if z(t\_{1}) = z\_{t\_{2}} only for t\_{1} = t\_{2}.
         6. Arc is a closed curve if the end points coincide : z(\alpha) = z(\beta).
            1. For closed curves a shift of the parameter is defined as follows.

If the orifinal equation is z = z(t), \alpha \le t \le \beta , we choose a point t\_{0} and define a new closed curve whose equation is z = z(t) for t\_{0} \le t \le \beta and z = z(t-\beta + \alpha) for \beta \le t \le t\_{0} + \beta - \alpha.

* + - 1. Opposite arc of z = z(t) , \alpha \le t \le \beta is the arc z = z(-t), -\beta \le t \le - \alpha.
      2. a constant function z(t) defines a point curve.
      3. A circle C can be considered as a close curve with the equation z = a + re^{it}, 0 \le t \le 2\pi. This is a standard parametrization for a circle.
    1. Analytic functions in Regions
       1. A complex-valued function f(z), defined on an open set \Omega, is said to be analytic in \Omega if it has a derivative at each point of \Omega.
          1. f(z) is said to be complex analytic, holomorphic
          2. The real and imaginary parts of an analytic function in \Omega satisfy the Cauchy-Riemann equations. Conversely, if u and v satisfies the Cauchy-Riemann equations and if the partial derivatives are continuous, then u + iv is an analytic function in \Omega.
       2. A function f(z) is analytic on an arbitrary point set A if it is the restriction to A of a function which is analytic in some open set containing A.
       3. Connected \Omega is said to be a region.
       4. Analytic function in \Omega degenerates if it reduces to constant.
       5. (Thm 3.11) An analytic function in a region \Omega whose derivative vanishes identically must reduce to a constant. the same is true if either the real part , the imaginary part, the modulus, or the argument is constant.
    2. Conformal Mapping
       1. Suppose that an arc \gamma with the equation z = z(t), \alpha \le t \le \beta is contained in a region \Omega, and let f (z) be defined and continuous in \Omega. then the equation w = w(t) = f(z(t)) defines an arc \gamma’ in the w-plane which may be called the image of \gamma.
          1. w’(t\_{0}) \neq 0. \gamma ‘ has a tangent at w\_{0} = f(z\_{0}), and its direction is determined by arg w’(t\_{0}) = arg f’(z\_{0}) + arg z;(t\_{0}).

Two curves which form an angle at z\_{0} are mapped upon curves forming the same angle. In view of this property the mapping w = f(z) is said to be conformal at all points with f’(z) \neq 0.